Directed cycles of length 4 in oriented bipartite graphs

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Abstract

In this note we obtain a new sufficient condition for the existence of directed cycles of length 4 in oriented bipartite graphs. As a corollary, a conjecture of H. Li (Rainbow C_3's and C_4's in edge-colored graphs, Discrete Math., to appear) is confirmed.

Keywords: Directed cycle; Oriented bipartite graph

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Throughout this note, we consider finite simple oriented graphs only, i.e., graphs without multiple edges and loops and in which each edge is replaced by only one arc. Let D be an oriented bipartite graph with bipartition (A, B). For $A_1 \subseteq A$ and $B_1 \subseteq B$, we denote by $A_D(A_1, B_1)$ the set of arcs from A_1 to B_1 in D. For terminologies and notations not defined here, we refer to Bondy and Murty [1].

In [4], G. Wang et al. raised the following conjecture.

Conjecture 1. Let D be a directed bipartite graph with bipartition (A, B). If $d^+(u) > \frac{|B|}{3}$ for $u \in A$ and $d^+(v) \geq \frac{|A|}{3}$ for $v \in B$, or $d^+(u) \geq \frac{|B|}{3}$ for $u \in A$ and $d^+(v) > \frac{|A|}{3}$ for $v \in B$, then there exists a directed C_4 in D.

G. Wang et al. [4] used a construction from [2] to show that if the conjecture holds, then it would be best possible. Now we rewrite the construction here. Let m and n be two positive integers divisible by 3. Let $|M_0| = |M_1| = |M_2| = \frac{m}{3}$ and $|N_0| = |N_1| = |N_2| = \frac{n}{3}$. We will construct an oriented bipartite graph with bipartition (M, N), where $M = M_0 \cup M_1 \cup M_2$ and $N = N_0 \cup N_1 \cup N_2$. Create all possible arcs from M_i to N_i, and from N_i to M_{i+1}, $i = 0, 1, 2$ (modulo 3). In the rest parts of this note, we use $D^*(m, n)$ to denote the construction above for convenience.

Recently, H. Li [3] proposed the following conjecture, which is a weak form of Conjecture 1. H. Li [3] proved the conjecture for balanced oriented bipartite graphs.

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Conjecture 2. Let D be an oriented bipartite graph with bipartition (A, B). If $d^+(u) > \frac{|B|}{3}$ for $u \in A$ and $d^+(v) > \frac{|A|}{3}$ for $v \in B$, then there exists a directed C_4 in D.

The main purpose of this note is to prove Conjecture 2. In fact, we prove a stronger result as follows.

Theorem 1. *Let D be an oriented bipartite graph with bipartition (A, B), where $|A| = m \geq 3$ and $|B| = n \geq 3$. If $d^+(u) \geq \frac{n}{3}$ for $u \in A$ and $d^+(v) \geq \frac{m}{3}$ for $v \in B$, then there exists a directed C_4 in D or $D = D^*(m, n)$.*

Proof. Let $D(m, n)$ be a family of digraphs consisting of all oriented bipartite graphs with bipartition (A, B) satisfying the conditions in Theorem 1, where $m = |A|$ and $n = |B|$. Choose any $D \in D(m, n)$.

First we claim that it is sufficient to prove Theorem 1 for those m and n which are both multiples of 3. Suppose that Theorem 1 holds for $D(m, n)$, where $m \equiv n \equiv 0 \pmod 3$. If $m \equiv n \equiv 0 \pmod 3$, then it is trivially true. Otherwise, without loss of generality, assume that m is not divided by 3. Let $s_1 = 3\lceil \frac{m}{3} \rceil - m$, $A^* = \{u_1, \cdots, u_{s_1}\}$ and $A' = A \cup A^*$. If $n \equiv 0 \pmod 3$, let $B' = B$. Otherwise, let $s_2 = 3\lceil \frac{n}{3} \rceil - n$, $B^* = \{v_1, \cdots, v_{s_2}\}$ and $B' = B \cup B^*$. Now we construct a new oriented bipartite graph D' with bipartition (A', B'), where $A(D') = A(D) \cup \{u'v, v'u : u \in A, u' \in A^*, v \in B, v' \in B^*\}$. Notice that $d^+_{D'}(u) \geq \lceil \frac{|B|}{3} \rceil = \frac{|B'|}{3}$ for each $u \in A$ and $d^+_{D'}(u') = n > \frac{|B'|}{3}$ for each $u' \in A^*$. Similarly, $d^+_{D'}(v) \geq \lceil \frac{|A|}{3} \rceil = \lceil \frac{|A'|}{3} \rceil$ for each $v \in B$ and $d^+_{D'}(v') = m > \frac{|B'|}{3}$ for each $v' \in B^*$. It follows that $D' \in D[3\lceil \frac{m}{3} \rceil, 3\lceil \frac{n}{3} \rceil]$. Hence there is a directed C_4 in D'. Since the vertices in A^* and B^* only have outdegrees, the directed C_4 in D' is also in D. The proof of our claim is complete.

Now we assume $m = 3m_1$ and $n = 3n_1$, where m_1, n_1 are two positive integers. Let D_1 be a spanning subdigraph of D satisfying $d^+_{D_1}(u) = n_1$ for $u \in A$ and $d^+_{D_1}(v) = m_1$ for $v \in B$. Suppose that there is no directed C_4 in D_1. Let u_0 be a vertex with maximum indegree k_1 among all the vertices in A, and v_0 be a vertex with maximum indegree k_2 among all the vertices in B. Let $B_1 = N^-_{D_1}(u_0)$, $B_2 = N^+_{D_1}(u_0)$, $A_3 = N^+_{D_1}(B_2)$ and $B_3 = N^+_{D_1}(A_3) - B_2$, where $|B_1| = k_1$, $|B_2| = n_1$. Since there is no directed C_4 in D_1, we have $N^+_{D_1}(A_3) \cap B_1 = \emptyset$. Since $|B_3|k_2 \geq |N^-_{D_1}(B_3)| \geq |A_{D_1}(A_3, B_3)| = |A_3|n_1 - |A_{D_1}(A_3, B_2)| \geq |A_3|n_1 - (|A_3||B_2| - |A_{D_1}(B_2, A_3)|) = |A_{D_1}(B_2, A_3)| = \frac{nm}{9}$, we get $|B_3| \geq \frac{nm}{9k_2}$. Therefore $|B| = n \geq |B_1| + |B_2| + |B_3| \geq k_1 + \frac{n}{3} + \frac{nm}{9k_2} \geq \sqrt[3]{\frac{k_1 n^2 m}{k_2}}$. It follows that $k_2 n \geq k_1 m$. By symmetry, we also have $k_1 m \geq k_2 n$. Thus, $k_1 m = k_2 n$ and all the inequalities above become equalities. This implies $|B_1| = |B_2| = |B_3| = n_1$, and $|A_{D_1}(B_2, A_3)| = |A_{D_1}(A_3, B_3)| = \frac{mn}{9}$. That is, $A_{D_1}(B_2, A_3) = \{vu : v \in B_2, u \in A_3\}$ and $A_{D_1}(A_3, B_3) = \{uv : u \in A_3, v \in B_3\}$. Notice that $|A_{D_1}(A_3, B_3)| = \frac{mn}{9} = |A_3| \cdot \frac{n}{3}$, we obtain $|A_3| = m_1$. Furthermore, we obtain the

fact $k_1 = \frac{n}{3}$ and $\sum_{u\in A} d^-_{D_1}(u) = \sum_{v\in B} d^+_{D_1}(v) = \frac{mn}{3}$. It follows that $d^-_{D_1}(u) = \frac{n}{3} = n_1$ for $u \in A$. Similarly, we have $d^-_{D_1}(v) = \frac{m}{3} = m_1$ for $v \in B$. Now let $A_1 = N^+_{D_1}(B_3)$ and $A_2 = N^-_{D_1}(B_2)$. Since there is no directed C_4 in D, $A_1 \cap A_2 = \emptyset$ and $A_{D_1}(A_1, B_2) = \emptyset$. Also we obtain $A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$. Notice that $|A_{D_1}(A_3, B_3)| = \frac{nm}{9}$ implies all inneighbors of any vertex of B_3 are in A_3. Hence $A_{D_1}(A_1, B_3) = \emptyset$ and $N^+_{D_1}(A_1) = B_1$, which means $A_{D_1}(A_1, B_1) = \{uv : u \in A_1, v \in B_1\}$ and $|A_1| = \frac{m}{3}$. Similarly, we have $A_{D_1}(B_1, A_2) = \{vu : v \in B_1, u \in A_2\}$, $A_{D_1}(A_2, B_2) = \{uv : u \in A_2, v \in B_2\}$, and $|A_2| = \frac{m}{3}$. Now we can easily deduce $D_1 = D^*(m, n)$. If there are other arcs in D but not in D_1, then obviously, there would be a directed C_4 in D, a contradiction. Thus $D = D_1 = D^*(m, n)$. The proof is complete. □

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